CHAPTER 2

Best Practice

"There is little we do in America that is more important than teaching. Effective teaching of mathematics requires appropriate pedagogical and mathematical foundations, but thrives only in an environment of trust which encourages leadership and innovation. In short, teaching must become more professional."

National Research Council, 1989.

Chapter 2

Introduction

One distinguishing characteristic of a profession is a body of specialized knowledge on which to base practice. It is only recently that the education profession has become more research-based in an attempt to inform and support teacher behavior and decision making. A research-based rationale for instructional practice is imperative if we are to counter the popular, but mistaken, notion that teaching requires no specialized knowledge (Clough, 1992).

"Pedagogical content knowledge differentiates expert teachers in a subject area from subject area experts."
Cochran, 1992

Shulman (1986), in a study of teaching, found that knowledgeable and skilled teachers possess and use a comprehensive and synthetic kind of professional knowledge. Shulman labeled this knowledge as "pedagogical content knowledge" which includes the special understandings and abilities that skilled teachers use in their efforts to help students understand complex ideas. Pedagogical content knowledge integrates content knowledge in the discipline with content knowledge about teaching. It also incorporates understanding of content, curriculum, learning, and teaching so that teachers can make effective decisions about learning outcomes, curriculum materials, teaching strategies, and assessment tasks.

Professional knowledge is a key component for teachers as they reexamine their beliefs and revise their practice to meet the reform challenges in mathematics education. The vision of reform and the belief that all students can and should become "mathematically powerful" is supported by the *Minnesota Graduation Standards* and the National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics*. Mathematical power includes the development of self-confidence and the ability to explore, conjecture, reason logically, and use a variety of mathematical methods to solve complicated real-world problems (NCTM, 1989, p. 5). Supporting mathematically powerful students requires:

- developmentally appropriate content based on important mathematics
- flexible instructional practices that encourage mindful and enduring learning by all students
- assessment practices that provide feedback on student progress and instructional effectiveness

"Teachers who begin to base their practice on [these] principles...should not expect to develop a finished repertoire of behaviors that, once achieved, will become routine. There is no point of arrival, but rather a path that leads to further growth and change."
Schifter, 1996

This chapter focuses on "best practice," the synthesis of research-based instructional strategies aimed at improving mathematics learning for all students. Central to best practice is research concerning how children learn mathematics. The chapter, therefore, begins with the theoretical foundation which makes the case for the importance of active, engaged learning and the necessity of aligning assessment with curriculum and instruction. The chapter then presents eight important best practice strategies. Though presented separately, the instructional strategies cited in this chapter are intended to be complementary, not mutually exclusive. Finally, to support the ultimate goal of improved mathematical learning and outcomes, the chapter ends with a section on reaching all students.

The information in this chapter is intended to broaden, but not to dictate, instructional practice. It describes some of the research on which best practice strategies are based and implications for instruction. This chapter also provides some jumping off points for looking critically at teacher practice, with the understanding that professional development and support are key to implementing instructional changes. It is hoped that the information on best practice in this chapter will encourage teachers to:

- form study groups to investigate best practice strategies in greater depth
- seek professional development opportunities to improve instructional strategies

" We must be impatient enough to take action and patient enough to sustain our efforts until we see results."
NCTM, 1996

- organize peer coaching and classroom observations to inform and support their implementation of best practice
- conduct research in their classrooms
- support change in the classroom, department, school, district, and system

This chapter does not cover all best practice strategies nor does it cover them in the depth or breadth of a thorough review. Resources that contain more information are listed at the end of each topic discussed. These were chosen because of their ability to provide rich and productive leads into educational research and literature as well as their accessibility. The list of cited references is included at the end of the chapter.

The chapter is organized as follows:

Part 1 · Focusing on the Learner

"Tell me, I forget.

Show me I remember.

Involve me, I understand."

Chinese proverb

Focusing on the Learner

Brain Research

" It is difficult to understand the brain because, unlike a computer, it was not built with specific purposes or principles of design in mind."
Fischbach, 1992

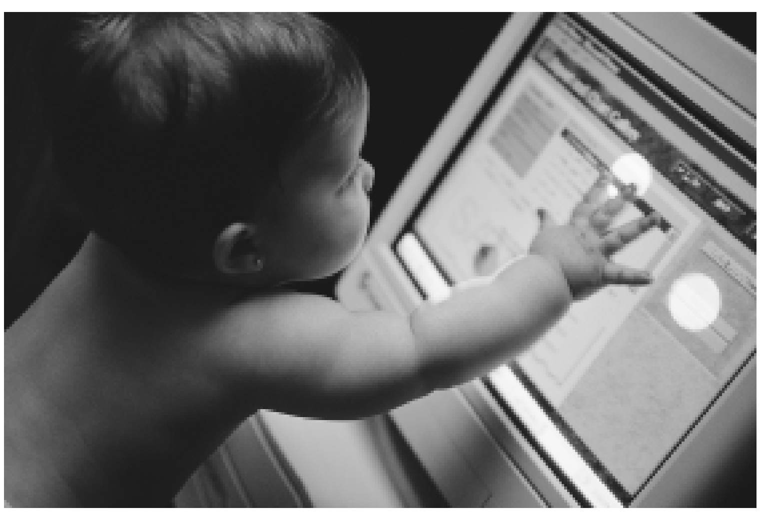
Developments in molecular biology, behavioral studies, endocrinology, and imaging technology have allowed researchers to literally see what is happening inside the brain while it works. This research, coupled with developments in cognitive psychology, has resulted in a growing interest in what is referred to as brain-based learning. This phrase is a way of pointing out and emphasizing that there should be a relationship between what we know about how the brain works and how children can learn best (Caine, 1997).

Not many years ago the brain was viewed as rule-bound, hard-wired, preset and fixed—more or less like a tape-recorder, storing whatever it heard, saw, tasted or felt. Research in the past decade has shown that, although the basic structural features of the brain are determined by genes, it is the environment that provides the necessary stimulation for the brain to achieve its potential. Nature and nurture must work together.

Brain research provides important information that can impact classroom instructional strategies:

- The brain is the ultimate hard-working athlete. The harder it exercises, or is used, the more "in shape" it appears to be as measured by the number of neurological connections found. Nerve connections (synapses) that are not strengthened by continuing stimulation from the environment die off.
- The brain is a pattern-seeker that requires external stimulation. Learning experiences should be designed to help learners identify patterns which facilitate connections with or modifications of previously learned patterns. Curriculum should be chosen with care and should emphasize the important interrelationships among the content and processes of mathematics.
- The brain connects emotion and cognition, therefore the classroom environment should be one that instills learners with confidence and is non-threatening. Work should be appropriately challenging and should teach students that they are not helpless, but rather are capable of exerting control over their learning.

" The brain is the ultimate 'use it or lose it' machine."
Kotulak, 1996



Cognitive Psychology Research

For most of this century, educational practice has been driven by behaviorism, a theory about how humans learn. Behaviorism assumes that students are blank slates (tabula rasa) on which the knowledge of others is simply written. Teaching based on behaviorism is characterized by:

- behavioral objectives
- tightly sequenced curricula
- drill and practice
- an emphasis on facts and principles
- an assumption that through mastering a series of simple steps, students are then able to engage in higher order thinking

Recent research in cognitive psychology has led to a theory of learning known as constructivism. Constructivism is deeply rooted in the work of a number of researchers, including Piaget, Dienes, and Bruner. It is the learning theory that forms the basis for instructional recommendations in this *Framework*.

"It is important to understand at the outset that constructivism is not an instructional approach; it is a theory about how learners come to know."
Airasian & Walsh, 1997

The central tenets of constructivism are these:

- Humans are not blank slates, but rather they are makers of meaning. Individuals learn by actively constructing and reconstructing their own knowledge about how the world works.
- Humans learn by modifying old ideas, not simply by accumulating new ones. Learning involves a give-and-take between what we know or believe about something and new information. This process is a continuous one.
- Learning is mental work—it isn't always fun. It often includes periods of feeling perplexed, frustrated, and sometimes even angry.
- New understandings are both personally and socially constructed or negotiated. Our social and cultural interactions influence the way we make sense of the natural world.
- While we can explain things to others, we cannot understand it for them.
- Students make their own sense out of new information whether we like it or not. "When students cannot readily assimilate new data into existing mental structures, they construct new relationships or schema in order to accommodate the new knowledge" (Stiff et al., 1993, p. 7).

Making Teaching Decisions Based on How Children Learn

Teachers must make decisions about the content and instructional strategies which best support student learning. Cognitive psychology provides the major theoretical rationale for the promotion of active student involvement in the learning process. It is at least as concerned with how children learn as with what they learn (Post, 1992, p. 20).

" The curriculum is a 'mind-altering device.'"
Eisner, 1997

- When learning is viewed as a continuing process of construction-reconstruction, learning tasks require an ongoing pattern of organization and reorganization. Curricula must deal directly with a few important ideas that are progressively sequenced.
- The construction of new knowledge takes time and sufficient experiences in a variety of contexts. This argues for a focused, coherent curriculum. Constructivism lends considerable support to the contemporary slogan that "less is more," the assertion that it is better to learn fewer things well than many things poorly.
- To make sense of the learning environment, students must be actively engaged. Learning must be "hands-on/minds-on."
- Because most learning is a social process, active involvement must be accompanied by students' interpretation and explanation of their findings. Students must have opportunities to explicitly share their ideas with other students, reflect on similarities and differences between these ideas, and revise their ideas and solutions as necessary.

Constructivist classrooms are not places where all ideas about mathematics are equal or where students vote on ideas and explanations. Constructivism requires teachers to intervene, not by saying to a student, "that's wrong," but by posing or presenting questions, activities, demonstrations, or investigations that help students reconsider their ideas as they develop and refine their understanding. In order for this to happen, a classroom environment must exist in which all members accept various strategies, encourage respectful listening, and welcome argument around mathematical understanding. In a constructivist classroom, teachers:

" In reality, no one can teach mathematics. Effective teachers are those who can stimulate students to learn mathematics."
National Research Council (NRC), 1989

- view students as thinkers with emerging theories about mathematics
- value and actively elicit student questions
- seek the students' points of view in order to assess their understanding
- rely heavily on primary sources of data
- generally behave in an interactive manner, mediating the environment for students

Providing Opportunities for Active Student Engagement

Brain research supports cognitive psychology theory and validates what good teachers have always known. Active engagement creates the brain activity fundamental to the learning process. Robust learning requires a change in the learner. Neurological connections are made and reinforced by what the learner does—what he or she attends to, the activities in which he or she engages.

"The learner must have experience with hypothesizing and predicting, manipulating objects, posing questions, researching answers, imagining, investigating and inventing."
Fosnot, 1993

In mathematics classes focused on providing a hands-on/minds-on environment, students:

- use mathematics to communicate their own thinking about complex situations with pictures, diagrams, graphs, words, symbols, and numerical examples
- solve problems using a variety of mathematical models and tools, such as manipulatives, calculators, and computers
- do projects and activities
- share and critique their own and others' mathematical ideas and products
- are thoughtful, persistent, flexible, self directed, and confident

The quality and durability of students' learning depends in part on the complexity of the tasks they encounter, the time students are given to make sense of their ideas, the flexibility of teachers in meeting student needs, and the connections that are made among equivalent representations of the same concept. Heibert (1996) reports that in classrooms emphasizing a conceptual approach to mathematics, tasks were chosen which make mathematics problematic for the student, connect with where students are mathematically, and leave students with something of mathematical value. Good tasks are ones that do not separate mathematical thinking from mathematical concepts or skills, that capture students' curiosity, and that invite them to speculate and to pursue their hunches (NCTM, 1991).

Mathematics is built from human activity—sorting, counting, ordering, comparing. From a young age children are naturally interested in mathematics and gradually develop a complex set of informal ideas about quantity in the natural environment. Effective instructional practices help learners reshape and internalize new information to make sense of the mathematics presented to them. As learners interact with new ideas and situations, they produce more complex knowledge based on their experiences and views of the world (Confrey, 1990, p. 109).

A major goal of both constructivism and the *Minnesota Graduation Standards* is to create autonomous learners. Students who come to know science and mathematics as a way of acting on, exploring, and understanding their world are more likely to be able to find a place in it that allows them to use their full capabilities. Classrooms must promote the abilities of students to initiate and sustain continuous inquiry. Such classrooms then become mathematical communities where students exchange ideas and learn concepts by talking, exploring, discovering, and thinking about their work.

Analyzing Student Learning to Improve Instruction

Active learning demands active assessment. "The emphasis on construction forces us to probe deeply into students' activity. How firm a grasp do they have on the material? What can they do with it? What misconceptions do they entertain? Even if they are producing wrong answers, are they constructing in a way that is mathematically recognizable? These are among the questions we need to ask in order to teach effectively" (Noddings, 1990, p 14). The Mathematical Sciences Education Board (MSEB, 1993, p. 1) presents three educational principles that form a foundation for assessment that can support effective education:

" In the constructivist approach, we look not for what students can repeat, but for what they can generate, demonstrate, and exhibit."
Brooks & Brooks, 1993

- Content—Assessment should reflect the mathematics that is most important for students to learn.
- Learning—Assessment should enhance mathematics learning and support good instructional practice.
- Equity—Assessment should support every student's opportunity to learn important mathematics.

Assessment is a communication process that tells teachers, parents, and students what is important to learn and what students already know and can do. Mathematics assessment must be aligned with curriculum and instruction to provide students with opportunities that are rich in breadth and depth and promote deep mathematical understanding.

Traditionally, mathematics instruction and assessment have been organized in ways that underestimate the mathematical capability of most students, thereby unintentionally filtering out students from further study of mathematics (NCTM, 1995, p. 1). Teachers need to find out what students know and think about a concept prior to selecting the context for instruction. Using careful observations and asking key questions while students interact with the problem at hand are important for guiding student thinking. Subsequent teaching can then move the child from his or her present level of understanding to the next.

During and following instruction, it is important to assess students' abilities to:

- use mathematical processes, such as computation, in the context of many kinds of problems rather than in isolation
- use mathematics to make sense of complex situations
- work on extended investigations
- formulate and refine hypotheses
- collect and organize information
- explain a concept orally or in writing
- solve problems that reflect those encountered in real life

All learning activities are not equally worthwhile. The same can be said of assessments. They should mirror real-life skills and knowledge, represent instructional practice, document what students know and can do, and provide feedback about the quality of curriculum, instruction, and achievement.

For Further Study of Topics in Part I

Brooks, J.G. & Brooks, M.G. (1993). *The case for constructivist classrooms*. Alexandria, VA: Association for Supervision and Curriculum Development.

Clements, D.H. & Battista, M.T. (1990). "Constructivist teaching and learning." *The Arithmetic Teacher*, 38(1), 34-35.

Kotulak, R. (1996). *Inside the brain*. Chicago, IL: Andrews and McMeel.

Webb, N. & Coxford, A. (Eds.). (1993). *Assessment in the mathematics classroom: 1993 yearbook*. Reston, VA: NCTM.

Part 2 · Promoting Effective Classroom Instruction and Assessment

"Learning mathematics requires construction, not passive reception, and to know mathematics requires constructive work with mathematics objects in a mathematical community."

Davis, Maher, & Noddings, 1990

" No other decision that teachers make has greater impact on students' opportunity to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages the students in studying mathematics. Here the teacher is the architect, the designer of the curriculum."

Lappan, 1993

Planning Meaningful Tasks

Background

Choosing meaningful learning experiences for students is one of the most important instructional decisions a teacher makes. The importance of worthwhile tasks is addressed in the *Professional Standards for Teaching Mathematics*. "The mathematics tasks in which students engage—projects, problems, constructions, applications, exercises, and so on—and the materials with which they work frame and focus students' opportunities for learning mathematics in school" (NCTM, 1991, p. 24).

Reys and Long (1995, p. 297) characterize good tasks as those that:

- are often authentic, coming from the students' environment
- are challenging, yet within reach of students
- pique and take advantage of students' curiosity
- encourage multiple perspectives and interrelated mathematical ideas
- nest skill development in the context of problem solving
- encourage students to make sense of mathematical ideas

What Research Says

" Students will not be provoked to inquire, learn, or study what they already know, or think they know, or what they consider at the moment to be irrelevant."

Rowe, 1978

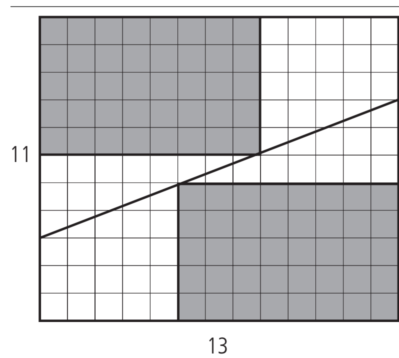
Support for problem-centered instruction can be found in a number of research studies including the work of Wood and Sellers (1997), who compared achievement of students receiving zero to two years of problem-centered instruction. Those in the two year group had "significantly higher scores on standardized achievement measures, better conceptual understanding, and more task-oriented beliefs for learning mathematics" (p. 163). Teachers can support student construction of mathematical understanding by embedding worthwhile tasks in a series of connected problems. Mathematics learning is enhanced when tasks provide students opportunities to use previously learned concepts and techniques in the process of discovering new ones (Lappan and Briars, 1995).

Choosing meaningful tasks involves two important components: the role of context in learning and the role of conceptual conflict in accommodating new knowledge (Rowe, 1978, p. 180). Context represents the situations in which activities are embedded. These become part of what is learned and how that learning is remembered and recalled (Lappan & Briars, 1995). Students who comment, "This is the same math we used to solve the Ferris Wheel problem!" are relating the mathematics to the context in which it was learned.

" The teacher needs to change the emphasis of problem solving from "Here's a problem, solve it" to "Here's a situation, let's explore it!"

Showalter, 1994

Conceptual conflict exists when new information does not seem to fit with what one already knows. When teachers produce situations in which there exists room for differences of interpretation, students will attempt to resolve those differences. The following example, An Area Paradox (Jacobs, 1970, p. 21), can create conceptual conflict for students which motivates them to investigate the mathematics in order to resolve the situation.



EX: Cut out the six pieces in the figure and throw the two shaded rectangular pieces away. The area of the remaining pieces is 63 square units. Now rearrange the four pieces to form a square. What is the area of this figure? Explain how this can be.

Implications for the Classroom

Designing or choosing tasks demands thoughtful and often collaborative planning on the teacher's part. A good mathematical investigation is one that not only invites finding one or more solutions but also allows for extensions beyond the immediate problem situation. Teachers can help facilitate such possibilities by scaffolding questions that allow students to continue reasoning through the problem or by asking challenging questions that motivate students to pursue problem extensions.

"The tasks in which students engage must encourage them to reason about mathematical ideas, to make connections, and to formulate, grapple with, and solve problems."
NCTM, 1991

The following list (Lappan et al., 1996, p. 40) can be used as a guide for evaluating classroom activities and reexamining what students are being asked to do in their mathematics textbooks. A rich problem solving task:

- has important, useful mathematics embedded in it
- may have different solutions or allow for different decisions or positions to be taken and defended
- can be approached by students in multiple ways using different solution strategies
- encourages student engagement and discourse
- requires higher level thinking and problem solving
- contributes to the conceptual development of students
- promotes the skillful use of mathematics
- creates opportunities for teachers to assess what their students are learning and where they are having difficulty

Engagement is necessary but not sufficient for mathematical learning. Student discussion and reflection must accompany active engagement to develop understanding of mathematical concepts. Also critical to concept development are the challenges inherent in the task, the materials and technology used to support the investigation, and student interaction focused on making sense of the mathematics. While students are engaged in problem-solving tasks, the teacher must listen to students' conversations and ask key questions. This provides the teacher with insights into levels of student understanding and further instructional needs.

For Further Study

Armstrong, B. (1995). "Teaching patterns, relationships, and multiplication as worthwhile mathematical tasks." *Teaching Children Mathematics*, 1(7), 446-450.

Showalter, M. (1994). "Using problems to implement the NCTM's professional teaching standards." *Mathematics Teacher*, 84(1), 5-7.

National Council of Teachers of Mathematics (NCTM). (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Using Concrete Materials

Background

"The child invents mathematical knowledge from her or his actions on objects, so direct, concrete experiences with many objects at the child's developmental level are crucial to the formation of accurate concepts."
Maxim, 1989

The NCTM *Professional Standards for Teaching Mathematics* recommend the use of concrete materials as one of many tools for enhancing communication and mathematical reasoning. Success with manipulatives depends greatly on the teacher's ability to choose appropriate objects and to create connections with the underlying mathematical concepts. Students do not always associate their work with concrete materials to the corresponding mathematical abstractions. Teachers can help students complete the process by asking probing questions and encouraging students to make these connections explicit.

When used wisely, manipulatives can:

- help make abstract ideas concrete
- lift mathematics off textbook pages
- help students construct connections between mathematical ideas, vocabulary, and symbols
- give students physical evidence to test and confirm their reasoning
- serve as concrete models for students to use to solve problems
- intrigue and motivate while helping students learn (Burns, 1996, p. 47)

What Research Says

According to research, explorations and investigations with manipulatives are an excellent way of providing students with tangible mathematical experience. Using manipulatives promotes active learning, models mathematics, and builds motivation. Studies indicate that the use of concrete objects is integrally related to the development of meaning. As children work with objects and talk about what they are doing, meaning is created and assimilated.

"Students who learn with manipulatives are better able to cross the bridge to the abstract world of mathematical concepts and apply their knowledge to real-life situations."
Kober, 1991

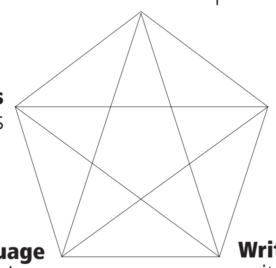
A review by Suydam and Higgins (Suydam, 1982) concerning manipulative use found that:

- lessons using manipulative materials are more likely to produce greater mathematical achievement
- the inclusion of the concrete stage in a sequence of instruction improves achievement
- studies at every grade level and across a variety of mathematical topics support the importance of the use of manipulative materials
- the use of materials appears to be effective with children at all achievement levels and all ability levels

Sowell (1989) concluded that long-term use of concrete materials increased student achievement more than short-term use. In addition, teacher training and knowledge of manipulative use was a key factor in influencing the effectiveness of instruction.

The research of Piaget, Dienes, and Bruner suggests that children's concepts evolve from direct interaction with the environment. Clements and McMillen (1996) capture this when they write, "Mathematical ideas are ultimately...integrated...not by their physical or real-world characteristics but rather by how 'meaningful'—connected to other ideas and situations—they are" (p. 271). Lesh (1979) found that using manipulatives in conjunction with pictorial, verbal, symbolic, and real-world representations can maximize learning. Connections to, and translations between, different representations are important cognitive processes which lead to a more robust understanding of concepts. Figure 1 shows the Lesh model applied to various representations of fractions.

Figure 1



Implications for the classroom

Effective instruction integrates experiences with concrete materials, which should be used:

- frequently
- primarily by students rather than by teachers for demonstration purposes
- in conjunction with other tools and conceptual representations
- in ways appropriate to the mathematical content under investigation
- in conjunction with exploratory and inductive instructional approaches
- as an aid in organizing content

"...allowing students to move the objects themselves is preferable to having a teacher demonstrate the action."
Kober, 1991

The scope of manipulative use in classrooms needs to be expanded. Tools such as computers, calculators, and measuring and drawing devices should complement the more familiar materials including geoboards, base ten blocks, and algebra tiles. Children need opportunities to use objects appropriate to the mathematical concepts being studied, the children's developmental levels, and their learning styles. Instruction should encourage children to use flexible and informal methods and avoid rote manipulation of materials.

"Teachers and students should avoid using manipulatives as an end—without careful thought—rather than as a means to that end."
Clements & McMillen, 1996

Not all types of manipulatives are right for all children, nor is one set of manipulatives appropriate for teaching all topics. However, worthwhile lessons, activities, or units that effectively use concrete materials engage both students' hands and minds. Careful use and sequencing of tasks involving manipulatives can help students create a bridge between concrete models and operations or abstract concepts. This leads to a deeper understanding of mathematics and requires students to develop evidence-based explanations for how things work.

For Further Study

Ball, D.L. (1992). "Magical hopes: Manipulatives and the reform of math education." *American Educator*, 16(2), 14-18, 46-47.

Clements, D. & McMillen, S. (1996) "Rethinking 'concrete manipulatives.'" *Teaching Children Mathematics*, 2(5), 270-279.

National Council for Supervision of Mathematics. (1994). "Suggestions on manipulatives." *Supporting leaders in mathematics education*. Golden, CO: Author. Also available on SciMath[MN] website: http://www.informns.k12.mn.us/scimathmn.

Integrating Technology

Background

"Increased use of technology in mathematics education is inevitable, but wise use is not automatic. Effective use of calculators and computers requires objectives for mathematics education that are aligned with the mathematical needs of the information age."
MSEB, 1990

Life in the 21st century will require personal competence with technology, information processing, communication, and decision making. With the integration of technology in every aspect of the workplace, the integration of technology in teaching and learning is essential. New, easy-to-use technologies offer increasingly versatile tools that can supplement and reinforce, not replace, student learning.

Used appropriately, new technology can enhance opportunities for children to engage in higher-order thinking. For example, the TI-92 graphing calculator and *Mathematica* and *Maple* software can perform the routine manipulation of symbols typically associated with the study of high school mathematics from algebra through calculus allowing students to focus on deeper structural elements of mathematics. *Geometer's Sketchpad* and *Cabri Geometry* are part of the growing number of dynamic geometry software programs that allow students to actively explore the interplay between shape, space, and measurement. Students can represent and manipulate geometric figures, investigate relationships among figures, and explore and test geometric conjectures. The Internet continues to provide new instructional and exploratory environments through interactive use of "notebooks," computer files specifically developed for mathematical explorations.

"Failure to introduce and to use calculators and computers in school creates a needless barrier between what is happening in students' everyday lives and what they are being taught in school."
Shane & Tabler, 1981

Swadener & Blubaugh (1990) believe that technology must be fully available to all students and teachers at all times as part of an instructional emphasis on problem solving skills and concept development. An NCTM position statement (1994) recommends that, "Teachers should use computers as tools to assist students with the exploration and discovery of concepts, with the transition from concrete experiences to abstract mathematical ideas, with the practice of skills, and with the process of problem solving."

Students who have not learned to use a calculator effectively to explore problems will be at a disadvantage in many testing situations. The Scholastic Aptitude Test (SAT) and American College Test (ACT) now endorse calculator usage while the Advanced Placement (AP) Calculus and Statistics tests are written so that calculator usage is necessary or advantageous.

What Research Says

"The major influence of technology...is its potential to shift from an emphasis on skills to an emphasis on developing concepts, relationships, structures, and problem-solving."
Corbitt, 1985

Research results indicate that calculator use for instruction and testing enhances learning and performance. This is true for arithmetical concepts and skills as well as problem solving (Hembree & Dessart, 1992). Research is consistent in showing that calculators do not have a negative impact on students' computational skills, including mastery of basic facts and proficiency in applying computational algorithms (Hembree & Dessart, 1986).

It is now clear that an understanding of arithmetic can be developed with a curriculum that emphasizes estimation, mental arithmetic, and calculator use, with reduced instruction in paper and pencil calculation. Indeed, there is evidence that overemphasis on manual skills hinders children's learning of when and how to use those skills (MSEB, 1990).

Researchers have also studied the effects of computer graphics on student understanding. Early studies of *Logo* and *Geometric Supposer* software suggest that students using exploratory programs may perform as well as or better than other students on traditional criteria (MSEB, 1990). For mathematical concepts, such as data analysis or functions, the computer seems to enhance student interest and understanding of important ideas. In addition, Demana and Waits (1990, p. 212) suggest that the greatest benefits appear to come from interactive technology that:

- is controlled by the user
- promotes student exploration
- enables generalization

Implications for the Classroom

Technology impacts the mathematics curriculum in three major ways (NCTM 1991):

- some mathematics becomes more important because technology requires it
- some mathematics becomes less important because technology replaces it
- some mathematics becomes possible because technology allows it

"Rather than replacing pencil and paper computation procedures, calculators are more likely to reinforce them; rather than substituting for independent thought, calculators are apt to sustain it."
Kober, 1991

Since computers and calculators can be invaluable tools to help students focus on generating ideas, trying out various approaches, and checking hunches, technology can help make mathematics accessible. If the context and challenge of a problem can be grasped by the students, then computation is no longer a barrier to solving the problem. The opportunity to use technology in doing mathematics allows students to:

- explore very complex problems and sophisticated concepts, with real data from real experiments and applications
- focus on strategies and test predictions with greater ease
- discuss and reflect on the mathematical principles behind the operations
- estimate, predict, share their discoveries, and question the reasonableness of their answers
- decide what type of technology tool/method is appropriate in a given situation
- learn the limitations of calculators, computers, and other technological tools

The unique instructional opportunities provided by calculators and computers support a change in teaching emphasis from getting the answer to reflecting on the process. No one today knows just what technology applications will be available in the next year or the next century. Instruction that emphasizes familiarity and flexibility with technological tools, conceptual understanding of mathematics, and facility with problem-solving approaches will be necessary to better equip students to deal with their future in an increasingly technological world.

For Further Study

Campbell, P.F. & Stewart, E.L. (1993). "Calculators and computers." In R.J. Jensen (Ed.), *Research ideas for the classroom: Early childhood mathematics*. Reston, VA: NCTM.

Fey, J.T. & Hirsch, C.R. (1992). *Calculators in mathematics education: 1992 yearbook*. Reston, VA: NCTM.

King, J. & Schattschneider, D. (Eds.). (1997). *Geometry turned on: Dynamic software in learning, teaching and research*. Washington, D.C.: The Mathematics Association of America.

"What we assess and how we assess it communicates what we value."
NCTM, 1995

"Assessment involves gathering evidence about a student's knowledge of, ability to use, and disposition toward mathematics and making inferences from that evidence for a variety of purposes."
NCTM, 1995

Assessing Student Performance

Background

Assessment is the process of describing what mathematics students know and can do. Some think that assessment, evaluation, and testing are synonymous. They are not.

Assessment is the process of gathering information. Once data have been collected, the diverse pieces of information are interpreted and integrated into a summary judgment. This is *evaluation*. *Tests* are measuring devices that are used to document student learning on narrowly defined questions and tasks. They are intended to produce a score in a more-or-less neutral and decontextualized environment. The tendency in the past has been to reduce assessment to testing, often using the data from a single test to indicate a student's capabilities. In reality, tests are just one of many tools available to the assessment process.

Because assessment is a feedback mechanism, it communicates teachers' expectations for students. For example, if communicating and defending a solution strategy to a mathematics problem is important, assessment should reflect this. On the other hand, when a student's knowledge of specific facts or algorithms is required, a multiple-choice or short answer test may be appropriate. Assessments must reflect the learning goals we have for our students as well as our instructional strategies.

What Research Says

Different aspects of achievement are measured by traditional tests and performance measures. For example, research indicates that students do not necessarily do better on performance tests than on other tests; they perform differently because the tasks are different. There is often a wide gap between the ability of students to demonstrate procedures and their ability to explain the procedures.

"Prior knowledge, experience, and the opportunity to learn are important considerations in interpreting test results."
NCTM, 1989

For these reasons, a number of meaningful tasks are required to be able to generalize about student performance and/or program effectiveness. Additionally, a variety of assessment formats are required to obtain a comprehensive view of student achievement. It is important to realize that performance tests are not equally interchangeable and each appears to measure different yet related aspects of mathematics achievement.

Questions of fairness and equity are as important in performance assessments as in other forms of assessment. Wording, topic or task selection, format, and scoring approaches can all introduce bias and influence performance.

Implications for the Classroom

Good assessment in mathematics:

- focuses on what students *know* and *can do* rather than what they *do not know*
- matches the curriculum in both *what is taught* and *how it is taught*
- is unbiased and fair for all students
- allows students to learn at various paces and honors various styles of learning and performance
- uses questions that require thoughtful responses
- compares performance over time to help recognize patterns of success and/or difficulty

Recent techniques collectively known as "alternative," "authentic," "active," and "performance" assessments reflect new understandings of how students learn. These assessments encompass a variety of methods, including: enhanced multiple-choice, teacher observations, checklists, investigations, portfolios, computer simulations, projects, and student self-assessments. Formats which allow for alternative ways to demonstrate understanding provide for individual differences. These differences may be the result of culture, gender, primary language, economic background, disabilities, or other factors that are masked in a single, traditional form of assessment.

"Assessments can contribute to students' opportunities to learn important mathematics only if they reflect, and are reinforced by, high expectations for every student."
MSEB, 1993

Rich learning activities can provide opportunities for informal assessment and can be modified to become performance assessment tasks. Professional educational journals and books also include examples and descriptions of teachers' experiences with authentic assessment. However, even with these resources, the process of developing assessments is often time-consuming and sometimes frustrating. Stenmark (1991, p. 1) makes the following recommendations for educators making changes in classroom assessment practice:

- don't try to do it all at once; start small but start somewhere
- don't try to do it all alone; find someone to work with, preferably at your grade level or within your discipline

The first step in developing assessment tasks is to identify the "big idea" of a unit. This is the "What is worth knowing?" question. Next, the criteria for judging student learning should be developed. Checklists are a useful way of displaying criteria and of clearly identifying the characteristics of a good product. Finally, students should have access to the checklists as they prepare for and complete an assessment. They can also use the checklists to make a judgment about the quality of their work that can be used as the basis for a discussion with them about their progress.

"In order to develop mathematical power in all students, assessment needs to support the continued mathematics learning of each student."
NCTM, 1995

Expanding the scope of assessment provides a broader view of student capabilities and knowledge. Teachers are then better prepared to describe and comment on each student's learning to parents and administrators, as well as to the students themselves. In addition, continual assessment of student understanding helps guide both long and short range instructional decisions designed to:

- ensure that every student is learning sound and significant mathematics
- support the development of a positive disposition toward mathematics
- challenge and extend student's ideas
- identify student needs in order to adapt or change activities

There is a dynamic interplay between instruction and assessment. Ideally, the lines between instruction and assessment become blurred when tests are part of instruction and instructional tasks are rich diagnostic opportunities.

For Further Study

Mathematical Sciences Education Board (MSEB). (1993). *Measuring what counts: A conceptual guide for mathematics assessment*. Washington, D.C.: National Academy Press.

National Council of Teachers of Mathematics. (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

Stenmark, J. (Ed.). (1991). *Mathematics assessment: Myths, models, good questions, and practical suggestions*. Reston, VA: NCTM.

Part 3 · Promoting Effective Classroom Interactions

"Teach all students as you would the brightest—with interdisciplinary curriculum, cooperative learning, hands-on activities, and a stronger focus on developing student's critical thinking skills."

Lynn & Wheelock, 1997

"Slowing down may be a way of speeding up."
Rowe, 1986

Increasing Wait Time

Background

In the late 1960s, Mary Budd Rowe analyzed instruction by teachers in a wide variety of settings. She found that teachers asked questions of students at the rate of two or three per minute. If student replies were not given within one second, the questions were repeated, rephrased, or answered by someone else. If students did respond quickly enough, the teacher then replied on average within nine-tenths of a second by asking another question or responding to the given answer (1978).

What Research Says

"Teachers who have learned to use silence report that children who do not ordinarily say much start talking and usually have exciting ideas."
Rowe, 1996

Rowe referred to the period of silence that follows teachers' questions as "wait time." She found that when wait times were increased to three seconds or longer the following aspects of children's and teachers' conversations increased as well:

- the length of student responses
- the number of unsolicited, relevant responses from students
- the number of student questions and the amount of speculative thinking
- student confidence
- the use of evidence in student responses
- the contributions by low-achieving students
- the creativity of responses

Increasing wait time also helped teachers to:

- ask more reflective and varied questions
- show more flexibility in responding to students
- decrease the total number of questions asked
- decrease disciplinary comments
- hold higher expectations for all students

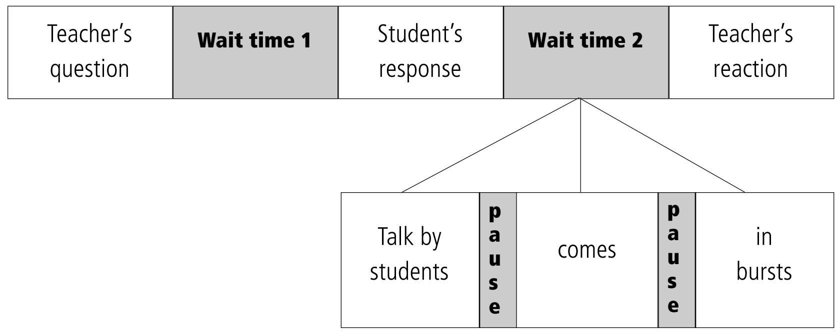
The research on wait time is very consistent. These findings hold across all disciplines as well as all levels, elementary through college.

Implications for the Classroom

There are several places during instruction when increasing wait time can have a positive effect on the level of discourse and thinking in the mathematics classroom:

- after the teacher asks an initial question and prior to calling on a student or group of students for a response (Wait time 1)
- after the response from a student or group of students to the initial question (Wait time 2)
- after receiving a student's question and prior to responding
- after asking a follow-up question

These opportunities can be identified in the following graphic (Rowe, 1978, p. 273):



"Wait time 2" is the term applied to the pause that follows a student's response. When a pause occurs here, the probability is higher that the students will continue a mathematical conversation among themselves.

Wait time is a key component in soliciting student responses to all forms of questions asked in class. Pausing has many purposes for the teacher and the students. For the teacher, pausing helps him/her focus on the student response and gives him/her time to understand the response and then to formulate appropriate follow-up questions. Additionally, if the teacher pauses, students are provided with time to understand the question or response and formulate their own thoughts, questions or elaborations. Equally important are the student pauses that indicate they are listening to other students' questions or responses. These wait times contribute to the growing of a community of mathematical thinkers rather than an audience dominated by a few fast thinkers.

"Optimal wait time may be dependent on the cognitive level of questions and the cognitive level of outcomes to be achieved."
Tobin, 1987

In practice, increasing wait time it is not always straightforward. There is a relationship between the amount of wait time and the level of the question asked. Lower level questions need shorter wait times than those requiring more thought from students. However, as wait time increases, the number of higher level cognitive questions that teachers ask increases as well. Since some students are ready to respond quickly, they should be acknowledged, but wait time should not be curtailed. This will give more students the opportunity to engage in mathematical thinking.

For Further Study

Rowe, M.B. (1986). "Wait time: Slowing down may be a way of speeding up." *Journal of Teacher Education*, 37(1), 43-50.

Rowe, M.B. (1996). "Science, silence and sanctions." *Science and Children*, 34(1), 35-37.

Tobin, K.G. (1987). "The role of wait time on higher level cognitive learning." *Review of Educational Research*, 57(1), 69-95.

"The notion that mathematics is a set of rules and formalisms invented by experts, which everyone else is to memorize and use to obtain unique, correct answers, must be changed."
Romberg, 1992

Questioning for Higher Order Thinking

Background

In the *Professional Standards for Teaching Mathematics*, NCTM (1991) indicates that classroom discourse, or "the ways of representing, thinking, talking, agreeing, and disagreeing" is central to helping students develop mathematical understanding and skills (p. 34). Teachers can "orchestrate discourse by posing questions and tasks that provoke and challenge students' thinking and by asking students to clarify and justify their ideas" (p. 37).

The development of higher order thinking cannot be achieved without teachers asking a variety of questions to challenge students' thinking—questions that require more than factual recall. Unfortunately, the research of Watson and Young found that teachers ask as many as 50,000 questions a year with at least 80 percent requiring only general recall of information (Vacc, 1993a, p.88).

In contrast, when teachers ask for explanations or follow-up responses, students have the opportunity to process and describe their own thinking. This not only provides necessary support for student learning but helps teachers assess student knowledge. Understanding student thinking provides necessary information for carefully planning follow-up questions and activities to move students' learning forward.

What Research Says

"Teachers' acceptance of student ideas is positively correlated with student learning gains."
Gall, 1984

Enough is known about questioning to establish that effective teacher questions contribute significantly to student learning. In their review, Prichard and Bingaman (1993, p. 220) report that achievement improves when students are asked more high level versus low level questions. Similarly, brain research findings show that those students who are asked to synthesize and report on their thinking increase their understanding and retention of new information by 75 percent (Wolfe, 1997). Student achievement increases for test questions that demand the use of higher cognitive level abilities. In general, student responses are more complex and at a higher cognitive level (Gabel, 1995, p. 127).

Research indicates that teachers must be conscious of their questioning practices in order to plan effective ways to stimulate and develop student thinking skills (Dantonio, 1990, p.16). Equally important to the level of the question is a well-designed sequence of questions with a focus on student responses.

Implications for the Classroom

"Improving the quality of teachers' questions is not sufficient. Students also need to learn the response requirement of different types of questions."
Gall, 1984

Several types of questions have the capacity to increase the cognitive level of student responses and shift the environment from one of "show and tell" to one of inquiry and discussion:

- Reasoning questions require students to construct logically organized information, e.g., "How do you know?", "What would happen if...?"
- Open questions allow for more than one acceptable answer, e.g., "Tell us everything you notice about...."
- Interpretive questions focus on applications, relationships, connections, or evaluations and lead students to analyze facts, e.g., "How would this be different if..."

In addition, certain techniques have proven valuable for increasing student participation and learning. *Think-pair-share* is a teaching strategy that gives students the opportunity to reflect individually before sharing their thinking with a partner or within a small group. The small group then shares its ideas with the whole group, listening to, paraphrasing, and comparing other group solutions to their own. This process complements the learning cycle, a method of organizing instruction that closely resembles the way people spontaneously construct knowledge (Lawson et al., 1989).

A *learning cycle* approach to instruction typically follows four stages. First, a teacher "launches" an investigation by introducing a rich task and asking questions that will pique student curiosity and motivate them. Next, students explore the problem alone, in pairs, or in small groups. The teacher then pulls the students back together to summarize their findings and attach labels/terms to the concepts involved in the lesson. Finally, students apply their learning to new situations.

"The authority for changing ideas comes from the results of experiments. Students have to learn to trust their ability to find answers. They have to feel safe in asking questions. They need time to think and a safe environment in which to speculate."
Rowe, 1978

Many new curriculum projects, in both mathematics and science, reinforce the learning cycle concept in their teacher support materials. The investigative nature of the tasks, supported by the questioning skills of the teacher, helps students develop and refine their thinking skills. Teachers are encouraged to pose questions which ensure worthwhile student activity and lead students to explore a concept or explain their thinking.

Facilitating questioning and thinking skills in the classroom is an art that, with effort, develops over time. The following strategies can help teachers improve the effectiveness of their questioning interactions with students:

- use precise language
- acknowledge all responses
- paraphrase student responses to acknowledge them
- rephrase questions rather than repeat them
- use non-specific praise sparingly
- acknowledge student performance by giving specific feedback
- ask students to "think about their thinking"
- encourage students to ask questions of you and other students
- consciously plan for productive interaction

For Further Study

Gall, M. (1984). "Synthesis of research on teachers' questioning." *Educational Leadership*, 42(3), 40-46.

Johnson, D.R. (1982). *Every minute counts: Making your math class work*. Palo Alto, CA: Dale Seymour Publications.

Vacc, N. (1993a). "Implementing the *Professional standards for teaching mathematics*: Questioning in the mathematics classroom." *Arithmetic Teacher*, 41(2), 88-91.

Structuring Collaborative Learning

" What students can do with others today, they can do alone tomorrow."
Vygotsky, 1962

Background

Quality teams, collaborative research teams, and project groups are common ways of organizing to get work done in the world outside school. There is a growing recognition that productive work groups require a variety of skills from their members, such as leading, following, listening, communicating, making decisions, negotiating, seeking feedback, and resolving conflicts.

Group learning has a long history in U.S. education and has always been a part of a skillful teacher's repertoire. Recent research and practice in cooperative learning stems from work by psychologist Morton Deutsch beginning in the late 1940s. Deutsch (1962) proposed three possible motivational goal structures for students: competitive, individualistic, and cooperative.

" Knowledge and skills are of no use if the student cannot apply them in cooperative interaction with other people."
Johnson et al., 1986

In competitive classrooms, students compete with one another to see who is best; their perception is that there are clear winners and losers. In classrooms that promote individualistic interactions, students work independently and the achievement of other students is irrelevant. In cooperative learning situations, student achievement is interdependent and individual students can achieve their learning goals only if other students also achieve their goals.

Cooperative learning differs from traditional small group instruction in that groups are structured to be heterogeneous, to have shared leadership, and to have both group and individual accountability for learning the material. Group tasks are developed to promote student dependence on each other in order to accomplish the learning goal. These factors provide incentives for students to help each other to learn (Slavin in Grouws, 1995).

In a typical cooperative learning situation in mathematics, students work on a problem in groups of two to five. They exchange views, discuss different approaches and solutions, and persuade each other of the soundness of their arguments. By explaining their ideas to others, students clarify their reasoning, order their thoughts, revise their strategies, and expand their conceptual understanding.

What Research Says

" Cooperative learning is a powerful tool for increasing self-confidence as a learner and for fostering true integration among diverse student populations."
Davidson, 1990a

The effectiveness of cooperative learning is well established in research. One research review reports that 72 percent of cooperative learning studies showed higher achievement for students involved in cooperative learning (Good et al., 1992). Reports on studies comparing the achievement of high-, middle-, and low-achieving students in competitive, individualistic, and cooperative learning situations show that cooperative learning experiences tend to produce higher results. This is true for "all ages, subject areas, and for tasks involving concept attainment, verbal problem solving, categorization, spatial problem solving, retention and memory, motor performance, and guessing-judging-predicting. For rote-decoding and correcting tasks, cooperation seems to be equally as effective as competitive and individualistic learning procedures" (Johnson et al., 1986, p. 24).

Cooperative group structures provide many learning opportunities that do not typically occur in traditional classrooms including opportunities to resolve conflicting points of view (Yackel, Cobb & Wood, 1991). They also result in greater self-confidence, better group relations, more cross-cultural integration, improved acceptance of mainstreamed children, and enhanced social skills (Kober, 1991).

Research suggests that flexible grouping that mixes students of different achievement levels, genders, and races/ethnicities better stimulates achievement gains for all students. Comparisons between high-achievers working in heterogeneous cooperative groups versus individualized set-

"The key lesson from research is to keep groups flexible and rearrange [them] periodically."
Kober, 1991

tings show that those in cooperative groups match or exceed their peers on traditional achievement scores. The cognitive processes involved in having to talk through and explain new material enhances scores on retention tests and promotes the development of higher quality reasoning strategies (Johnson et al., 1986).

Implications for the classroom

Teachers establish the guidelines and expectations for working cooperatively, and must directly teach group processing and interpersonal skills. Kober (1991, p. 14) suggests that teachers play a critical role in collaborative learning by forming groups, observing and interacting with groups, answering and clarifying questions, and moderating and helping students to tie ideas together.

Johnson and Johnson (1991, pp. 279-281) suggest the following requirements for effectively structured cooperative lessons:

- positive interdependence—students must recognize that they need one another to complete the task, often summarized as, "We sink or swim together."
- face to face promotive interaction among group members
- interpersonal and small-group skills
- reflection on group processes
- individual accountability

One of the concerns in the use of groups is the so-called "hitchhiker" problem where certain students do the majority of the work. Teachers who spend time explaining the reasons for cooperative group work and who do not grade on a curve encounter the hitchhiker problem less often.

While cooperative learning is well-suited to a variety of instructional purposes, tasks that require multiple abilities and contributions for goal completion are likely to promote better cooperative activity and collaboration by all students in a group (Cohen et al., 1982).

Research suggests that, for all students, cooperative, heterogeneous, and flexible groupings for instruction are more effective for stimulating and improving achievement than the traditional independent learner approach. Slavin (in Yackel et al., 1990) suggests that the teacher is instrumental in structuring "a pervasive norm in the classroom that helping one's peers to learn is not a marginal activity, but is a central element of students' roles" (p. 20).

For Further Study

Davidson, N. (Ed.). (1990a). *Cooperative learning in mathematics: A handbook for teachers*. Menlo Park, CA: Addison Wesley.

Davidson, N. (1990b). "Small-group cooperative learning in mathematics" In T.J. Cooney (Ed.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.

Johnson, D.W. & Johnson, R.T. (1990). "Social skills for successful group work." *Educational Leadership*, 47(4), 29-33.

Incorporating Oral and Written Communication

Background

Learning is predominately a social process. It involves checking against personal experience and negotiation with peers and teachers. Constructing meaning is an active process that includes hands-on learning with the materials and tools of mathematics and a focus on interpretation and explanation of student findings. What was once referred to as discussion must now become a classroom conversation among peers. Ideas are shared, respected, and available for reflection, discussion, and revision.

"It boils down to this—if you can't talk about math, you are unlikely to do it well."
NCSM, 1997

Oral and written communications influence students' reasoning ability, construction of mathematical knowledge, problem-solving abilities, self-confidence, and social skills acquisition (Lappan & Schram, 1989). Group discussions encourage students to apply previously-learned knowledge to new situations. Performing a writing task requires students to reflect on, analyze, and synthesize the material being studied in a thoughtful and precise way (Davison & Pearce, 1988). Conversations and writing exercises help students identify gaps in their own understanding and support students' construction of knowledge.

Communication is also a necessary component of assessment. Listening to students explain their thinking provides information for planning follow-up questions and activities. Through student writing, teachers have access to a unique source of information which is typically underutilized as an assessment resource in mathematics classrooms. By responding to student communications, either individually or before the whole class, teachers engage in a unique and continuous dialogue that can contribute to the whole process of teaching and learning (Miller, 1992).

"Teachers who listen to students, and who plan instruction based on what they learn from listening, transform student learning."
NCSM, 1997

What Research Says

Research indicates that student communication positively influences learning. Piagetian research shows that children develop language and logical thinking ability through an exchange of viewpoints. Studies also show that time spent in content-oriented interactions with peers and teachers enhances classroom performance and achievement. As students synthesize their thoughts to share them with or teach another student, their ability to retain new information improves by 75 percent (Wolfe, 1997).

In studies where levels of student talk were structured and measured, increased levels of "on task" student talk were related to increased achievement. Achievement was significantly higher for groups in which student talk was equal to or half as much as teacher talk compared to groups with no student talk (Dessart & Suydam, 1983).

"The nature of the activity and talk in the classroom shapes each student's opportunities to learn about particular topics as well as to develop their abilities to reason and communicate about those topics."
NCTM, 1991

Research studies on problem solving have shown that expository writing is an effective and practical tool for enhancing learning (Miller, 1991). Improved mastery of mathematics concepts and skills is possible if students are asked to write about their understanding.

Implications for the Classroom

Facilitating emerging mathematical ideas and conversations requires skill and patience. Responsibility for initiating discussion and maintaining focus is generally assumed by the teacher, especially with younger children. Thus it becomes very important that teachers strengthen their skills in facilitating discourse.

"The teacher's role is to create an environment where students feel free to:

- *share their beliefs and opinions*
- *ask what, how, and why questions*
- *take risks*
- *hypothesize*
- *make mistakes"*

Vacc, 1993b

Teachers can reinforce classroom communication skills and help students understand the value of focused writing and conversation when they:

- teach what is expected by sharing the norms and routines of communication
- pose questions and tasks that elicit, engage, and challenge each student's thinking
- listen carefully to students' ideas
- arrange seating so that students can easily see classmates as they speak
- encourage and monitor each student's communication and participation

Creating environments in which students can safely communicate their own mathematical thinking is a central teaching strategy and a crucial element in developing students' mathematical power. Students learn about the nature of mathematical knowledge as they justify their choices from among different strategies and solutions. As their ideas become the focus of critical examination, a different view of mathematics is created—a view that mathematics is dynamic, growing, and created by people.

Promoting classroom discourse often demands that teachers think quickly on their feet and make minute by minute decisions regarding:

- when to join a group quietly without comment
- when to elaborate, provide information, clarify an issue, or model a problem
- when to let a student/group struggle with a difficulty
- when and how to attach appropriate mathematical notation and language to students' ideas

Students in the classroom should be involved in communication in which they:

- clarify and justify their ideas orally and in writing
- think about a focus question individually before discussing it in class
- provide an audience for their peers, speaking to, questioning, and listening to one another
- write or discuss ideas with a partner before sharing with the whole group
- seek clarification when they don't understand a question or statement

Students learn mathematics by talking, writing, reasoning, and reflecting about mathematics. They are able to develop proficiency in the language of mathematics through active use of that language in meaningful contexts (Santiago & Spanos, 1993, p. 134).

Mathematics classrooms must promote students' ability to ask questions, share ideas, and communicate thinking in a dynamic environment of learning. Students who talk and write about mathematics as a way of making sense of the world are more likely to be able to use their own questions in the future to direct their learning and their work.

For Further Study

Lappan, G. & Schram, P.W. (1989). "Communication and reasoning: Critical dimensions of sense making in mathematics." In P. Trafton & A. Schulte (Eds.), *New directions for elementary school mathematics: 1989 yearbook.* Reston, VA: NCTM.

Mokros, J., Russell, S.J., & Economopoulos, K. (1995). *Beyond arithmetic: Changing mathematics in the elementary classroom*. Palo Alto, CA: Dale Seymour Publications.

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

Part 4 · Reaching All Students

"Perhaps most important, is a fundamental underlying belief...Virtually everyone 'has it' in mathematics, and the job of the teacher (or school) is to help the student realize that potential."

Usiskin, 1993

Overview

"Equity for all requires excellence for all; both thrive when expectations are high."
NRC, 1989

An effective society respects and attends to the concerns of individuals and of groups. "Excellence has to do with individuals, and equity is a property of groups. Individuals and groups, like excellence and equity, ought always to be examined together. One without the other is incomplete" (Willie, 1997).

Excellence and equity are complementary goals—each one rounds out the other, and together they push the educational achievement bar higher for all students. To establish high standards and raise expectations without adequately addressing the needs of students in underserved populations is to perform an injustice first and foremost to those students, and just as importantly, to our society as a whole. The National Commission on Excellence in Education recognized this when they stated that favoring either goal alone could lead to "a generalized accommodation to mediocrity in our society on the one hand or the creation of an undemocratic elitism on the other" (*Nation at Risk* cited in Willie, 1997).

"Besides being an issue of justice, creation of a society where equal opportunity exists in access to mathematics has become an economic necessity."
NCTM, 1996

Although children enter school with the capability to learn mathematics, certain groups become underrepresented in mathematics classes and fail to fulfill their mathematical potential (Kober, 1991, p. 22). Existing disparities would suggest that education has not provided equal opportunity or equal access, nor has equal performance been attained. In tracked systems, low level mathematics classes are not effective in providing the powerful remediation required to maintain student progress through a standards-based curriculum. The slogan, "Mathematics for All," is a reminder of the imbalance that exists between equity and excellence in the educational arena.

In the SciMath[MN] *Statement of Equity for Mathematics and Science Education* (1997), equity means "equitable access to high-quality science and mathematics education and equitable treatment in classrooms, schools, and post-secondary education institutions for every student, leading to high levels of knowledge, skill, and educational attainment in mathematics and science. Transforming instruction in mathematics and science to achieve an equitable education system depends on the commitment and participation of all stakeholders, including students, families, business/industry partners, community members, policy makers, and educators at all levels. Working collaboratively, these stakeholders can ensure that an educational system provides challenging learning opportunities leading to high achievement in science and mathematics for ALL students." (See Chapter 7: *Making It Happen* for more specific information on what various stakeholders can do to support this effort; see Appendix C for the complete *SciMath[MN] Statement of Equity*.)

"It is clearly documented that inequities exist and that they can be alleviated."
Carey et al., 1995

Equitable instruction in mathematics must be geared toward inclusiveness with respect to gender, socioeconomic status, ethnicities/cultures, disabilities, and languages. In order to address equity issues in the mathematics classroom, teachers will need to investigate their own beliefs about "mathematics for all." Teacher awareness of equitable practices, self-observation via videotapes and/or peer coaching, and reflection with thoughtful action can begin to create classrooms which expand, rather than limit, the learning potential of all students.

Research suggests that groups, as well as individuals within groups, may have different learning needs according to their culture, language, socialization patterns, and expertise. Different teaching strategies may be more effective in meeting the diverse strengths and mathematical potential of some students. However, in traditional classrooms, direct instruction, individualized seat work, and memorization of procedures have dominated the teacher's instructional practice. Achieving a significant transformation in the mathematical learning experiences of all students will require aligning best practice with a coherent and focused curriculum and multiple and varied assessments.

For Further Study of Topics in Part 4

Armstrong, T. (1994). *Multiple intelligences in the classroom*. Alexandria, VA: Association for Supervision and Curriculum Development.

Cuevas, G. & Driscoll, M. (1993). *Reaching all students with mathematics*. Reston, VA: NCTM.

Lawrence, G. (1991). *People types and tiger stripes: A practical guide to learning styles*. Gainesville, FL: Center for Applications of Psychological Type, Inc.

Lynn, L. & Wheelock, A. (1997). "Making detracking work." *Harvard Education Letter*, 13(1).

Secada, W.G., Fennema, E., & Adajian, L.B. (1995). *New directions for equity in mathematics education*. Cambridge, UK: Cambridge University Press.

Thornton, C. & Bley, N. (1994). *Windows of opportunity: Mathematics for students with special needs*. Reston, VA: NCTM.

Trentacosta, J. & Kenney, M.J. (Eds.) (1997). *Multicultural and gender equity in the mathematics classroom: The gift of diversity: 1997 yearbook*. Reston, VA: NCTM.

"Justice in mathematics will not be achieved until the goals of education are met equally by both sexes."
Fennema, 1990

Gender

What Research Says

Research reviews by Jacobs and Becker (1997, p. 108) show that traditional teaching strategies in mathematics classes favor learning styles of boys over girls. This imbalance can lead to inequities in participation and outcomes.

A review of research shows both achievement and attitude differences between males and females:

- In high stakes tests, such as the mathematics subtest of the SAT, large gaps persist, with girls scoring 35 points lower than boys (National Coalition for Women and Girls in Education, 1996).
- Women are less likely to enter fields that require advanced mathematics and science degrees. While women hold 32 percent of bachelor degrees in computer/mathematics sciences, they hold only 18 percent of the doctoral degrees (NSF, 1996).
- In Minnesota, when comparing the number of baccalaureate and graduate degrees in mathematics, science, engineering, and technology, females only achieve parity with males in the life sciences (Johnson, 1997).

"There is no evidence that females as a group have less aptitude for math than males."
Kober, 1991

Research cited in Campbell (1995, p. 226) also indicates the following differences:

- Females are more apt than males to doubt their competence in mathematics and to be less confident in their mathematics ability.
- Self confidence drops for females in early adolescence before they experience any academic decline.
- Males receive more praise, a greater number of disciplinary contacts, and more general teacher-initiated contacts than females.

Implications for the Classroom

Mathematics teaching has traditionally stressed and emphasized deductive proof, absolute truth, certainty, algorithms, abstraction, logic, and rigor. To build on the strengths of all students, teachers must balance teaching strategies which address different learner needs. Teachers need to provide a balance of learning strategies by also promoting intuition, experience, conjecture, generalization, induction, creativity, and context in the mathematics classroom (Jacobs & Becker, 1997, p. 108).

In creating a gender-fair program, teachers can make a conscious effort to equalize the amount and type of classroom interactions which encourage female participation and success. They can:

"Gender differences in mathematics performance are predominantly due to the accumulated effects of sex-role stereotypes in family, school, and society."
NRC, 1989

- observe and monitor their questioning patterns to be sure they have equitable instructional interactions with both girls and boys
- eliminate both blaming and non-specific praising statements from their verbal behavior
- provide hands-on and/or spatial activities on a regular basis
- encourage cooperative interaction and discussion among students
- pair girls with girls often to provide increased opportunities for girls to use mathematics materials, technology, and assume leadership roles
- increase interactions with and expectations of females on high-cognitive-level mathematics activities
- accept student responses and encourage divergent thinking
- invite female mathematicians and scientists to class as role models and/or study female mathematicians.

" Research shows that students' cultural milieu greatly influences their learning preferences; people are socialized to learn in various ways."
NREL, 1997

Race, Ethnicity, and Culture

What Research Says

Culture is meaningful because it suggests what we can take for granted as common information, language, and priorities (Hilliard, 1995, p. 100). If teachers never employ styles that are compatible with cultural norms, students' connections with the mathematics curriculum and subsequent learning of mathematical concepts are greatly decreased (Stiff, 1990, p. 155).

Studies of students from underrepresented racial/ethnic groups show that:

- Performance gaps between white students and students of color are documented by the end of second grade, and are wider for older students and on questions requiring higher level thinking (Stiff, 1990).
- Large differences in performance and course enrollments exist; e.g., 60 percent of white and Asian students enroll in Algebra II versus less than 50 percent of African Americans, Hispanics and Native Americans (NSF, 1996, p. xv).
- African Americans take about one year less of high school mathematics than the national average (Kober, 1991).
- Non-white students are more heavily enrolled in lower-level math courses; e.g., 24-35 percent of Hispanics, African Americans, and Native Americans are enrolled in remedial classes versus 15 percent of whites and Asians (NSF, 1996, p. xv).
- NAEP results for Minnesota show that many ethnic groups perform below comparable groups from other states in spite of the high mean performance shown for Minnesota students in general (Reese et al., 1996).
- African Americans, Hispanics and Native Americans make up 12, 10 and 1 percent of the population respectively but form only 3.5, 3 and .02 percent of the science and engineering work-force (NSF, 1996, p. xviii).

Implications for the Classroom

" Attending school regularly is the most consistent and reliable predictor of success on the [Minnesota 8th grade] mathematics and reading examinations."
Myers, 1997

There are numerous examples of schools and classrooms where students exceed expectations for performance. The question, therefore, is not whether students of various racial/ethnic backgrounds can or cannot achieve high mathematical skills; rather it is which means will elicit maximum success in mathematics? (Walker & Chappell, 1997, p. 202). Teachers must:

- expect students of all racial/ethnic/cultural backgrounds to achieve
- consider students' language, culture, and community as assets rather than liabilities
- recognize that all racial/ethnic/cultural groups are part of traditions that have contributed to our common mathematics knowledge base
- increase student-to-student and student-to-teacher interactions
- increase the cognitive level of interactions with students of color
- avoid tracking, which systematically removes students from opportunities to learn important mathematics
- use diverse and flexible assessments to determine students' strengths
- use immediate and effective remediation to counteract poor performance results
- vary the instructional styles in the classroom

" Sí se puede, 'It can be done.'"
Flores, 1997

Language

What Research Says

A student's native language plays a role in the development of his/her mathematical skills and concepts. Research shows that concepts and skills can be reinforced when students have the opportunity to discuss them in their native language (Cuevas, 1990, p. 160). Flores (1997), however, reports that in many schools limited English proficiency (LEP) acts as a filter which limits students' opportunities in the following ways:

- limited English proficiency is often equated with limited talent and expertise
- a disproportionate number of children who do not speak English are labeled as learning disabled
- students are often prevented from discussing and explaining material to one another in their own language
- very few programs for gifted and talented students are offered in languages other than English

" Words and phrases have definite meanings and represent ideas that if not understood prevent the learner from fully understanding the message conveyed."
Cuevas, 1990

Students with limited English proficiency may also have specific difficulty with the precise use of language in mathematics compared to daily communication. The vocabulary, syntax, and format of mathematics may be unfamiliar to them. This can also hinder some learning disabled students who have difficulty with decoding, visual-spatial relations, directionality, and sequencing skills (Kober, 1991).

Language skills also include the components of listening, comprehending, reading, writing, and speaking (Cuevas, 1990, p. 159). Difficulties may be caused by inabilities to understand either the spoken or written word or both. Additionally, students may also exhibit differences between their conversational and their academic skill in either English or their native language.

Implications for the Classroom

Mathematics teachers can address LEP students' special needs by using the following strategies (Kober, 1991; Cuevas, 1990):

- recognize that not all LEP students have the same needs
- become familiar with each student's background in order to use culturally relevant problems and contexts
- modify lessons to cut down on the use of new vocabulary terms and phrases
- use visual aids, diagrams, and hands-on experiences to help clarify the meaning of verbal and written information
- incorporate activities that teach the language of mathematics in context; include vocabulary, listening, reading, speaking, and writing skills
- maximize communication opportunities for students in English as well as their native language
- adjust instruction, class discussions, and assessment to include visual representations and nonverbal communications with verbal instructions
- promote student enrollment in mathematics courses, since language development is attained better when language is used in context

" It is by integrating the development of language in the context of mathematics instruction that we may give these students the 'key that unlocks the door to the world of mathematics.'"
Cuevas, 1990

"With as many as 10 percent of the population disabled in some way, the nation can ill afford continued under-representation of disabled persons in mathematical careers."
NRC, 1989

Physical Disabilities

What Research Says

Most students with physical disabilities are quite capable of learning and achieving at the same level as their peers even though they may work more slowly or require adaptive devices. In spite of this, research shows that students with disabilities often have low academic self images.

Research focused on students with physical disabilities also indicates that (Kober, 1991; NRC, 1989):

- programs which place special needs students in separate classrooms negatively affect achievement
- inclusive programs benefit students academically and socially
- educational technology, including computers and calculators, can be adapted for use by disabled students and can result in higher mathematics achievement

"It is important to remember that students with special learning needs are individuals with many of the same needs as other students."
Bley, 1994

Implications for the Classroom

The teacher can assess the impact of a student's physical disability on learning by answering the following questions (Bley, 1994, p. 138):

- What learning strengths are intact?
- What learning abilities are suppressed by each disability?
- What learning or teaching styles are most compatible with this student's specific learning needs?

Structuring an appropriate mathematics education for the physically handicapped child may be a matter of providing or adapting appropriate instructional aids. In addition, teachers may need to:

- recognize that students with disabilities may have different life experiences, so some contextual references may not be as helpful to them
- assess the learning strengths of disabled students and teach to these strengths as often as possible
- use a variety of appropriate learning approaches, including visual, auditory, kinesthetic/tactile
- use technology to enhance learning experiences of student with disabilities
- use a partner or assistant to work with disabled students in the classroom as appropriate
- modify testing procedures and formats
- recognize the accomplishments of physically handicapped mathematicians and scientists

"In recent years, the growing use of computers as an aid for persons with disabilities and as a tool for mathematicians provides yet another effective link to enable persons with disabilities to succeed in mathematics-based careers."
NRC, 1989

Hearing impaired students may not appear to understand the mathematics when in fact they are not understanding the language that is being used (Bley, 1994, p. 142). When working with hearing-impaired students, teachers should highlight and write out new vocabulary and ideas using explanations that are direct, succinctly stated, and carefully presented.

Learning Styles

"There are no guarantees that an awareness of style will improve mathematics instruction, but there are some exciting possibilities."
Driscoll, 1980

What Research Says

The examination of learning styles has interested psychologists much longer than it has educators. This helps explain the fact that more is known about what learning styles are than what teachers should know or do about them (Driscoll, 1980). Howard Gardner's research (1993) on children and brain-damaged adults led him to point out that a "one size fits all" definition of intelligence is inadequate to describe a child's potential. This led him to develop a theory of multiple intelligences which states that individuals have different strengths in a range of intelligence areas, and that no one set of teaching strategies will work best for all students at all times.

"Learning style...is crucial in explaining why certain instruction works with some students and not with others"
Lawrence, 1991

Research has identified some effects of individual learning styles in the classroom:

- when students experience only one type of teaching style, some are at an advantage and others are at a disadvantage
- when teacher and student styles match, students are more likely to receive higher grades and evaluations than when they do not match
- while cultural experiences influence student learning styles, it should not be assumed that all members of any group have the same learning preference
- there is a tendency to think others are more like us than they really are and that others share the same learning styles
- styles are not as rigid as commonly believed and can vary depending on factors such as time, place, and task

"One clear conclusion of research is that educators must avoid the trap of assuming that a child's lack of success is due to a lack of talent. It could be due to a lack of opportunity or to an improper match between instruction and student learning style" (Driscoll 1980, p. 84).

Implications for the Classroom

Our own learning preferences reveal some of the differences that can affect mathematics learning and performance:

- sense modalities (visual, auditory, kinesthetic)
- preferences for working alone or with others
- linear/analytic approaches compared to global/spatial approaches
- reflective thinking versus impulsive doing

"In the classroom, the teacher continually shifts her methods of presentation from linguistic to spatial to musical and so on, often combining intelligences in a creative way."
Armstrong, 1994

Students who are aware of their own learning styles can be proactive in finding ways to complement their learning. In the classroom, the key to accommodating students' various learning styles is to thoughtfully and systematically vary the instructional approach of lessons. This can be done by:

- listening to students or using inventories to assess how students learn best
- using observations about student learning styles to modify instruction, not to label
- encouraging a variety of thinking styles in problem solving situations
- providing a variety of open-ended as well as structured learning situations
- providing a variety of materials
- varying the amount and type of practice to meet individual needs

" MYTH: Learning mathematics requires special ability, which most students do not have."
NRC, 1989

Detracking

What Research Says

In the United States many people believe that learning mathematics depends on special ability. In many other countries, however, students, parents, and teachers all expect that by working hard, most students can master mathematics. Student achievement in these countries and in effective intervention programs in the United States indicate that students achieve, not because of special techniques, but because they are exposed to consistent high-quality instruction and high expectations.

" One thing was certain: The children could not learn algebra if they were not exposed to it."
Hilliard, 1995

As a result of tracking practices based on perceived ability differences, many U.S. students do not have the opportunity to participate in standards-based mathematics learning. Research (Oakes, 1985, p. 91) shows that tracking:

- does not increase student learning
- prevents the opportunity to learn important mathematics content and increases the educational gap among students
- tends to widen the achievement gap and retards the academic progress of many students, especially those in the average and/or low achievement group (Barquet, 1992)
- results in an unfair and disproportionate placement of poor students and students of particular racial/ethnic groups in low achievement and non-college bound classes

" 'Twice as much, twice as fast, twice as hard' is not an appropriate program for highly talented students."
Keynes in NRC, 1989

Implications in the Classroom

Giving students the opportunity to learn mathematics means that teachers start where students are, not where "they ought to be." From that starting point, teachers expect and help students to go as far as possible. Accommodations for low and high achievers can still be used to provide remediation, enrichment, differential assignments, and depth of coverage. The goal, however, is to provide all students with the opportunity to study a core curriculum and make progress toward high standards.

Programs that are successful in addressing varying abilitities (Eddins & House, 1994, p. 321; Leder, 1995, p. 217):

- base instruction on problems and activities that can be approached from different levels (concrete to more abstract)
- ask open-ended questions that allow for individual exploration and investigation
- frequently assess student understanding and flexibly regroup students for different instructional units
- create a classroom environment that supports all students—an environment in which the teaching style mirrors the nature of mathematical inquiry
- promote a broader definition of what constitutes evidence of mathematical accomplishment
- involve multiple measures to identify and encourage promising students
- offer flexible pathways along which gifted students can encounter rich ideas through challenging, nonstandard learning experiences
- excite students about the wonder of mathematics and encourage them to invest their talents in mathematics
- expose students in lower achieving groups to more challenging content and sophisticated discussions about mathematics
- choose and/or modify tasks appropriate to the skills and the sophistication of understanding of individual students

" The best time to learn mathematics is when it is first taught; the best way to teach mathematics is to teach it well the first time."
NRC, 1989

" Homogenized is only better for milk."
Davidson & Hammerman, 1993

Reaching All Students: A Summary

In theory, homogeneous groups make it easier for teachers to help students who are all "at the same level." In reality, people don't come in homogeneous groups regardless of how hard we try to make them do so. "Even if we could find an entire class of children who tested at exactly the same level in mathematics, there would be some who got that score because they were intuitive problem solvers, others because they were fast at computations and therefore got a lot done, and still others because they were slow but very accurate in their work" (Davidson & Hammerman, 1993).

" The fundamental objective of education always has been to prepare students for life. The new objectives for school mathematics...do not depart from this tradition but rather, reaffirm it."
NCTM, 1996

The Curriculum and Evaluation Standards for School Mathematics and the *Minnesota Graduation Standards* clearly agree that all students can learn mathematics and must be given the opportunity to do so. This commitment to mathematics for all must effect changes in teachers' beliefs, curriculum materials, instructional strategies and assessment paradigms. "The goal of every teacher should be to provide all students with the opportunity to develop their full human potential in an environment where...differences are respected and valued and where full participation and partnership are the norm" (Stiff, 1993, p. 6).

There are no quick fixes to our present situation in school mathematics, but there are recommendations that can help us reflect and modify our classroom practice:

- Start from where students are.
  - Review only when necessary.
  - Place students in courses in which they have opportunity to be successful.
  - Allow students of different ages to do the same mathematics.
- Set high expectations.

  Students with talent in mathematics can be found in any group. Their potential will not develop, however, unless it is properly nurtured and encouraged.
- Provide the same high-quality curriculum and instruction for all students.

  Incorporate problem solving and real-world applications into the core curriculum.
- Provide multiple points of entry to rich mathematical tasks.

  Conceptual learning should not be put on hold while students develop proficiency in memorizing facts or computational speed.
- Emphasize all aspects of mathematical discourse.

  The nature of discourse and the level of interaction in the mathematics classroom is an important indication of how well the needs of different students are being met.
- Provide immediate and effective remediation.

  Make every attempt to prevent students from falling behind. Many students need to know that they are behind and, with support and encouragement, need to work harder to catch up.
- Use technology to support student learning of mathematics.

  Calculators and computers have the potential to make learning more active and dynamic, and hence, more effective.
- Use fair and meaningful assessment and testing procedures.

" Not only must the mathematics we teach meet the social and economic demands of [an increasingly complex] world, but also we must take steps to ensure that everyone participates in that world."
Secada, 1990

Providing both an excellent and an equitable mathematics education for all students will depend on the commitment and participation of all stakeholders, including students, families, business/industry partners, community members, policy makers, and educators at all levels. Working collaboratively, these stakeholders can ensure that our educational system provides challenging learning opportunities leading to high achievement in mathematics for ALL students.

References

Airasian, P.W. & Walsh, M.E. (1997). "Constructivist cautions." *Phi Delta Kappan*, 78(6), 444-449.

Armstrong, B. (1995). "Teaching patterns, relationships, and multiplication as worthwhile mathematical tasks." *Teaching Children Mathematics*, 1(7), 446-450.

Armstrong, T. (1994). *Multiple intelligences in the classroom*. Alexandria, VA: Association for Supervision and Curriculum Development.

Ball, D.L. (1992). "Magical hopes: Manipulatives and the reform of math education." *American Educator*, 16(2), 14-18, 46-47.

Battista, M.T. (1994). "Calculators and computers: Tools for mathematical exploration and empowerment." *Arithmetic Teacher*, 41(7), 412-417.

Baroody, A.J. (1993). *Problem solving, reasoning, and communicating, K-8: Helping children think mathematically*. New York, NY: Macmillan Publishing Company.

Barquet, N. (1994). "Excellence and equity: What research says about tracking." In National Council of Supervisors of Mathematics (Eds.), *Supporting leaders in mathematics education*. Golden, CO: NCSM.

Bley, N.S. (1994). "Accommodating special needs." In C. Thornton & N. Bley (Eds.), *Windows of opportunity: Mathematics for students with special needs*. Reston, VA: NCTM.

Brooks, J.G. & Brooks, M.G. (1993). *The case for constructivist classrooms*. Alexandria, VA: Association for Supervision and Curriculum Development.

Burns, M. (1996). "How to make the most of math manipulatives." *Instructor*, 105(7), 45-51.

Caine, R.N. & Caine, G. (1997). *Education on the edge of possibility*. Alexandria, VA: Association for Supervision and Curriculum Development.

Campbell, P.F. (1995). "Redefining the 'girl problem' in mathematics." In W. Secada, E. Fennema, & L. Adajian (Eds.), *New directions for equity in mathematics education*. Cambridge, UK: Cambridge University Press.

Campbell, P.F. & Stewart, E.L. (1993). "Calculators and computers." In R.J. Jensen (Ed.), *Research ideas for the classroom: Early childhood mathematics*. Reston, VA: NCTM.

Carey, D., Fennema, E., Carpenter, T., & Franke, M. (1995). "Equity and mathematics education." In W. Secada, E. Fennema, & L. Adajian (Eds.), *New directions for equity in mathematics education*. Cambridge, UK: Cambridge University Press.

Clements, D.H. & Battista, M.T. (1990). "Constructivist teaching and learning." *Arithmetic Teacher*, 38(1), 34-35.

Clements, D. & McMillen, S. (1996). "Rethinking 'concrete manipulatives.'" *Teaching Children Mathematics*, 2(5), 270-279.

Clough, M. (1992). "Research *is* required reading." *The Science Teacher*, 59(7), 36-39.

Cochran, K.F. (1992). "Pedagogical content knowledge: Teachers' transformations of subject matter." In F. Lawrenz, K. Cochran, J. Krajcik, & P. Simpson (Eds.), *Research matters...to the science teacher. NARST Monograph 5*. Manhattan, KS: Center for Science Education, Manhattan State University.

Cohen, P.A., Kulik, J.A., & Kulik, C. (1982). "Educational outcomes of tutoring: A meta-analysis of findings." *American Educational Research Journal*, 19(2), 237-248.

Confrey, J. (1990). "What constructivism implies for teaching." *Journal for Research in Mathematics Education*, Monograph Number 4, pp. 107-122.

Corbitt, M.K. (1985). "The impact of computing technology on school mathematics: Report of an NCTM conference. *Mathematics Teacher*, 78(4), 243-250.

Cuevas, G. (1990). "Increasing the achievement and participation of language minority students in mathematics education." In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.

Cuevas, G. & Driscoll, M. (1993). *Reaching all students with mathematics*. Reston, VA: NCTM.

Dantonio, M. (1990). *How can we create thinkers?* Bloomington, IN: National Educational Service.

Davidson, E. & Hammerman, J. (1993). "Homogenized is only better for milk." In G. Cuevas & M. Driscoll (Eds.), *Reaching all students with mathematics*. Reston, VA: NCTM.

Davidson, N. (Ed.). (1990a). *Cooperative learning in mathematics: A handbook for teachers*. Menlo Park, CA: Addison Wesley.

Davidson, N. (1990b). "Small-group cooperative learning in mathematics." In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.

Davis, R., Maher, C., & Noddings, N. (1990). "Constructivist views on the teaching and learning of mathematics." *Journal for Research in Mathematics Education*, Monograph Number 4.

Davison, D. & Pearce, D. (1988). "Using writing to reinforce mathematics instruction." *Arithmetic Teacher*, 35(8), 42-45.

Demana, F. & Waits, B.K. (1990). "Enhancing mathematics teaching and learning through technology." In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.

Dessart, D. & Suydam, M. (1983). *Classroom ideas from research on secondary school mathematics*. Reston VA: NCTM.

Deutsch, M. (1962). "Cooperation and trust: Some theoretical notes." In M. Jones (Ed.), *Nebraska Symposium on Motivation*. Lincoln, NE: University of Nebraska Press.

Downs, R.E., Matthew, J.L., & McKinney, M.L. (1994). "Issues of identification." In C. Thornton & N. Bley (Eds.), *Windows of opportunity: Mathematics for students with special needs*. Reston, VA: NCTM.

Driscoll, M.J. (1980). *Research within reach: Elementary school mathematics*. St. Louis, MO: CEMREL, Inc.

Eddins, S. & House, P. (1994). "Flexible pathways: Guiding the development of talented students." In C. Thornton & N. Bley (Eds.), *Windows of opportunity: Mathematics for students with special needs*. Reston, VA: NCTM.

Eisner, E.W. (1997). "Cognition and representation." *Phi Delta Kappan*, 78(5), 349-353.

Fennema, E. (1990). "Justice, equity, and mathematics education." In E. Fennema & G. Leder (Eds.), *Mathematics and gender*. New York, NY: Teachers College Press.

Fey, J.T. & Hirsch, C.R. (Eds.), (1992). *Calculators in mathematics education: 1992 yearbook.* Reston, VA: NCTM.

Fischbach, G.D. (1992, September). Mind and brain. *Scientific American*, 48-58.

Flores, A. (1997). "Sí se puede, 'It can be done': Quality mathematics in more than one language." In J. Trentacosta & M.J. Kenney (Eds.), *Multicultural and gender equity in the mathematics classroom: The gift of diversity: 1997 yearbook*. Reston, VA: NCTM.

Fosnot, C.T. (1993). "Rethinking science education: A defense of Piagetian constructivism." *Journal of Research in Science Teaching*, 30, 1189-1201.

Gabel, D. (1995). "Science." In G. Cawelti (Ed.), *Handbook of research on improving student achievement*. Arlington, VA: Educational Research Service.

Gall, M. (1984). "Synthesis of research on teachers' questioning." *Educational Leadership*, 42(3), 40-46.

Gardner, H. (1993). *Frames of mind: The theory of multiple intelligences*. New York, NY: Basic Books.

Good, T.L., Mulryan, C., & McCaslin, M. (1992). "Grouping for instruction in mathematics: A call for programmatic research on small-group process." In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York, NY: Macmillan Publishing Company.

Grouws, D.A. (1995). "Mathematics." In G. Cawelti (Ed.), *Handbook of research on improving student achievement*. Arlington, VA: Educational Research Service.

Hembree, R. & Dessart, D.J. (1992). "Research on calculators in mathematics education." In J.T. Fey & C.R. Hirsch, (Eds.), *Calculators in mathematics education: 1992 yearbook*. Reston, VA: NCTM.

Hembree, R. & Dessart, D.J. (1986). "Effects of hand-held calculators in pre-college mathematics education: A meta-analysis." *Journal for Research in Mathematics Education*, 17, 83-99.

Hiebert, J. (1996). Conference presentation: AERA SIGME pre-session. San Diego, CA.

Hilliard, A.G., III. (1995). "Mathematics excellence for cultural 'minority' students: What is the problem?" In I. Carl (Ed.), *Prospects for school mathematics*. Reston, VA: NCTM.

Jacobs, H.R. (1970). *Mathematics: A human endeavor*. San Francisco, CA: W.H. Freeman and Company.

Jacobs, J.E. & Becker, J.R. (1997). "Creating a gender-equitable multicultural classroom using feminist pedagogy." In J. Trentacosta & M.J. Kenney (Eds.), *Multicultural and gender equity in the mathematics classroom: The gift of diversity: 1997 yearbook*. Reston, VA: NCTM.

Jensen, R.J. & Williams, B.S. (1993). "Technology: Implications for middle grades mathematics." In D.T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics*. Reston, VA: NCTM.

Johnson, D. & Johnson, R. (1990). "Social skills for successful group work." *Educational Leadership*, 47(4), 29-33.

Johnson, D. & Johnson, R. (1991). "Classroom instruction and cooperative learning." In H. Waxman & H. Walberg (Eds.), *Effective teaching: Current research.* Berkeley, CA: McCutchan Publishing Corporation.

Johnson, D., Johnson, R., & Holubec, E. (1986). *Circles of learning*. Edina, MN: Interaction Book Company.

Johnson, D.R. (1982). *Every minute counts: Making your math class work*. Palo Alto, CA: Dale Seymour Publications.

Johnson, R.C. (1996). *SciMath^{MN} report on equity issues in science and mathematics education in Minnesota*. St. Paul, MN: SciMath^{MN}.

King, J. & Schattschneider, D. (Eds.). (1997). *Geometry turned on: Dynamic software in learning, teaching and research*. Washington, D.C.: Mathematics Association of America.

Kober, N. (1991). *Ed Talk: What we know about mathematics teaching and learning*. Washington, D.C.: Council for Educational Development and Research.

Kotulak, R. (1996). *Inside the brain*. Chicago, IL: Andrews and McMeel.

Lappan, G. (1993). "Implementing the *Professional standards for teaching mathematics*: What do we have and where do we go from here?" *Arithmetic Teacher*, 40(9), 524-26.

Lappan, G. & Briars, D. (1995). "How should mathematics be taught?" In I.M. Carl (Ed.), *Prospects for school mathematics*. Reston, VA: NCTM.

Lappan, G. & Schram, P.W. (1989). "Communication and reasoning: Critical dimensions of sense making in mathematics." In P. Trafton & A. Schulte (Eds.), *New directions for elementary school mathematics: 1989 yearbook*. Reston, VA: NCTM.

Lappan, G., Fey, J.T., Fitzgerald, W.M., Friel, S.N., & Phillips, E.D. (1996). *Getting to know CMP: An introduction to the Connected Mathematics Project*. Palo Alto, CA: Dale Seymour Publications.

Lawrence, G. (1991). *People types and tiger stripes: A practical guide to learning styles*. Gainesville, FL: Center for Applications of Psychological Type, Inc.

Lawson, A.E., Abraham, M.R., & Renner, J. (1989). "A theory of instruction: Using the learning cycle to teach science concepts and thinking skills." *Monograph of the National Association for Research in Science Teaching*, Cincinnati, OH: NARST.

Leder, G.C. (1995). "Equity inside the mathematics classroom: Fact or artifact?" In W. Secada, E. Fennema, & L. Adajian (Eds.), *New directions for equity in mathematics education*. Cambridge, UK: Cambridge University Press.

Lesh, R. (1979). "Mathematical learning disabilities: Consideration for identification, diagnosis, and remediation." In R. Lesh, D. Mierkiewicz, & M.G. Kantowski (Eds.), *Applied mathematical problem solving*. Columbus, OH: ERIC/SMEAR.

Lynn, L. & Wheelock, A. (1997). "Making detracking work." *Harvard Education Letter*, 13(1).

Mathematical Sciences Education Board (MSEB). (1990). *Reshaping school mathematics: A philosophy and framework for curriculum*. Washington, D.C.: National Academy Press.

Mathematical Sciences Education Board (MSEB). (1993). *Measuring what counts: A conceptual guide for mathematics assessment*. Washington, D.C.: National Academy Press.

Maxim, G.W. (1989). "Developing preschool mathematical concepts." *Arithmetic Teacher*, 37(4), 36-41.

Miller, L.D. (1991). "Writing to learn mathematics." *Mathematics Teacher*, 84(7), 516-521.

Miller, L.D. (1992). "Teacher benefits from using impromptu writing prompts in algebra classes." *Journal for Research in Mathematics Education*, 23(4), 329-340.

Mokros, J., Russell, S.J., & Economopoulos, K. (1995). *Beyond arithmetic: Changing mathematics in the elementary classroom*. Palo Alto, CA: Dale Seymour Publications.

Myers, Jr., S.L. (1997). *Analysis of the 1996 Minnesota Basic Standards Test Data*. Minneapolis, MN: Roy Wilkins Center for Human Relations and Social Justice.

National Coalition for Women and Girls in Education. (1997). *Title IX at 25: Report card on gender equity*. Washington, D.C.: Author.

National Council of Supervisors of Mathematics (NCSM). (1994). "Suggestions on manipulatives." *Supporting leaders in mathematics education*. Golden, CO: Author. Also available on SciMath^MN website: http://www.informns.k12.mn.us/scimathmn.html.

National Council of Supervisors of Mathematics (NCSM). (1997). *Supporting improvement in mathematics education*. Golden, CO: Author.

National Council of Teachers of Mathematics (NCTM). (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (1991). *Professional standards for teaching mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (1994). *Position statement: The use of technology in the learning and teaching of mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (1995). *Assessment standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics (NCTM). (1996). *A perspective on reform in mathematics and science education*, K.E. Vos (Ed.). Columbus, OH: Eisenhower National Clearinghouse for Mathematics and Science Education.

National Research Council (NRC). (1989). *Everybody counts: A report to the nation on the future of mathematics education*. Washington, D.C.: National Academy of Sciences.

National Science Foundation (NSF). (1996). *Women, minorities, and persons with disabilities in science and engineering: 1996*. Arlington, VA: NSF 96-311.

Noddings, N. (1990). "Constructivism in mathematics education." *Journal for Research in Mathematics Education,* Monograph Number 4, pp. 7-18.

Northwest Regional Educational Laboratory (NREL). (1997). *Science and mathematics for all students: It's just good teaching*. Portland, OR: Author.

Oakes, J. (1985). *Keeping track: How schools structure inequality*. New Haven, CT: Yale University Press.

Post, T.R. (1992). "Some notes on the nature of mathematics learning." In T.R. Post (Ed.), *Teaching mathematics in grades K-8: Research based methods*. Needham Heights, MA: Allyn and Bacon.

Prichard, M.K. & Bingaman, S. (1993). "Instructional activities and decisions." In P.S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics.* Reston, VA: NCTM.

Reese, C.M., Miller, K.E., Mazzeo, J., & Dossey, J.A. (1997). *NAEP 1996 mathematics report card for the nation and the states*. Washington, D.C.: National Center for Education Statistics.

Reys, B. & Long, V. (1995). "Implementing the *Professional standards for teaching mathematics:* Teacher as architect of mathematical tasks." *Teaching Children Mathematics*, 1(5), 296-299.

Romberg, T. (1992). "Further thoughts on the standards: A reaction to Apple." *Journal for Research in Mathematics Education*, 23(5), 432-437.

Rowe, M.B. (1978). *Teaching science as continuous inquiry: A basic*. New York, NY: McGraw Hill Book Company.

Rowe, M.B. (1986). "Wait time: Slowing down may be way of speeding up." *Journal of Teacher Education*, 37(1), 43-50.

Rowe, M.B. (1996). "Science, silence and sanctions." *Science and Children*, 34(1), 35-37.

Santiago, F. & Spanos, G. (1993). "Meeting the NCTM communication standards for all students." In G. Cuevas & M. Driscoll (Eds.), *Reaching all students with mathematics*. Reston, VA: NCTM.

Schifter, D. (1996). "A constructivist perspective on teaching and learning mathematics." *Phi Delta Kappan*, 77(7), 492-499.

Schmidt, W.H., McKnight, C.C., & Raizen, S.A. (1997). *A splintered vision: An investigation of U.S. science and mathematics education*. Boston, MA: Kluwer Academic Press.

SciMath[MN]. (1997). *Statement of equity for mathematics and science*. St. Paul, MN: Author.

Secada, W.G. (1990). "The challenges of a changing world for mathematics education." In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.

Secada, W.G., Fennema, E., & Adajian, L.B. (1995). *New directions for equity in mathematics education*. Cambridge, UK: Cambridge University Press.

Shane, H.G. & Tabler, M.B. (1981). *Educating for a new millennium: Views of 132 international scholars*. Bloomington, IN: Phi Delta Kappa Educational Foundation.

Showalter, M. (1994). "Using problems to implement the NCTM's professional teaching standards." *Mathematics Teacher*, 84(1), 5-7.

Shulman, L.S. (1986). "Those who understand: Knowledge growth in teaching." *Educational Researcher*, 15(2), 4-14.

Sowell, E. (1989). "Effects of manipulative materials in mathematics instruction." *Journal for Research in Mathematics Education*, 20, 498-505.

Stenmark, J. (Ed.). (1991). *Mathematics assessment. Myths, models, good questions, and practical suggestions*. Reston, VA: NCTM.

Stiff, L. (1990). "African-American students and the promise of the *Curriculum and evaluation standards*." In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.

Stiff, L. (1993). "Reaching all students: A vision of learning mathematics." In G. Cuevas & M. Driscoll, M. (Eds.), *Reaching all students with mathematics*. Reston, VA: NCTM.

Stiff, L.V., Johnson, J.L., & Johnson, M.R. (1993). "Cognitive issues in mathematics education." In P.S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics*. Reston, VA: NCTM.

Suydam, M.N. (1982). *Using manipulative materials to learn mathematics*. Columbus, OH: ERIC/SMEAC Mathematics Education Fact Sheet Number 2.

Swadener, M. & Blubaugh, W. (1990). "Teaching mathematics with a vision: Integrating computers into instruction." In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.

Thornton, C. & Bley, N. (1994). *Windows of opportunity: Mathematics for students with special needs*. Reston, VA: NCTM.

Tobin, K.G. (1987). "The role of wait time on higher level cognitive learning." *Review of Educational Research*, 57(1), 69-95.

Trentacosta, J. & Kenney, M.J. (Eds.). (1997). *Multicultural and gender equity in the mathematics classroom: The gift of diversity: 1997 yearbook*. Reston, VA: NCTM.

U.S. Department of Education, National Center for Education Statistics. (1996). *Pursuing excellence*. Washington, D.C.: U.S. Government Printing Office: NCES 97-198.

Usiskin, Z. (1993). "If everybody counts, why do so few survive?" In G. Cuevas & M. Driscoll, M. (Eds.), *Reaching all students with mathematics*. Reston, VA: NCTM.

Vacc, N.N. (1993a). "Implementing the *Professional standards for teaching mathematics*: Questioning in the mathematics classroom." *Arithmetic Teacher*, 41(2), 88-91.

Vacc, N.N. (1993b). "Implementing the *Professional standards for teaching mathematics*: Teaching and learning mathematics through classroom discussion." *Arithmetic Teacher*, 41(4), 225-227.

Vygotsky, L.S. (1962). *Thought and language*. Cambridge, MA: MIT Press.

Walker, P. & Chappell, M. (1997). "Reshaping perspectives on teaching mathematics in diverse urban schools." In J. Trentacosta & M.J. Kenney (Eds.), *Multicultural and gender equity in the mathematics classroom: The gift of diversity: 1997 yearbook*. Reston, VA: NCTM.

Webb, N. & Coxford, A. (Eds.). (1993). *Assessment in the mathematics classroom: 1993 Yearbook*. Reston, VA: NCTM.

Willie, C.V. (1997). "Can equity and excellence coexist?" *Education Week*, 17(7).

Wolfe, Pat. (1997). Presentation, Minnesota Superintendents Meeting. Bloomington, MN.

Wood, T. & Sellers, P. (1997). "Deepening the analysis: Longitudinal assessment of a problem-centered mathematics program." *Journal for Research in Mathematics Education*, 28(2), 163-186.

Yackel, E., Cobb, P., & Wood, T. (1991). "Small-group interactions as a source of learning opportunities in second-grade mathematics." *Journal for Research in Mathematics Education*, 22(5), 390-408.

Yackel, E., Cobb, P., Wood, T., Wheatley, G., & Merkel, G. (1990). "The importance of social interaction in children's construction of mathematical knowledge." In T.J. Cooney & C.R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990's: 1990 yearbook*. Reston, VA: NCTM.